CATEGORY THEORY
TOPIC II: COLLECTIONS

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1. Collections of Sets

We do not disallow the possibility that a set may be an element of another set. In fact, this idea is very useful. For example, we may talk about the set of lines in a plane, even though each line is a set of points in the plane. The set of lines is a set of subsets of the points in the plane. It is common to call sets whose elements are subsets of a given set a *collection* of subsets.

Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. Then the *intersection* and *union* of the sets in the collection are defined by

- $\cap\mathcal{C} = \{x \in X \mid x \in C \text{ for all } C \in \mathcal{C}\}$;
- $\cup\mathcal{C} = \{x \in X \mid x \in C \text{ for some } C \in \mathcal{C}\}$.

Thus $\cap\mathcal{C}$ is the intersection of all the sets in $\mathcal{C}$ and $\cup\mathcal{C}$ is their union.

Example 1. Let $A = \{n \in \mathbb{N} \mid n < 25\}$, $O = \{n \in A \mid n \text{ is odd}\}$, $P = \{n \in A \mid n \text{ is prime}\}$, and $S = \{n \in A \mid n \text{ is a square}\}$. Let $\mathcal{C} = \{O, P, S\}$. Then

- $\cap\mathcal{C} = \varnothing$, because no square is a prime;
- $\cup\mathcal{C} = \{2, 3, 4, 5, 7, 9, 11, 13, 15, 16, 17, 19, 21, 23\}$.

□

Example 2. Let $A = \{n \in \mathbb{N} \mid n < 1000\}$. For each $d \in \mathbb{N}$, define

$$D_d = \{n \in A \mid n = dm \text{ for some } m \in \mathbb{N}\}.$$

Let $\mathcal{D} = \{D_p \mid p \text{ is prime and } p \leq 7\}$. Find $\cap\mathcal{D}$.

Solution. The set D_d is the set of positive multiples of d which are less then 1000. The set $\mathcal{D}$ is the collection of all D_p such that p is a prime which is less than 7. Thus $\mathcal{D} = \{D_2, D_3, D_5, D_7\}$. Then $\cap\mathcal{D}$, being the intersection of these sets, is the set of natural numbers less than 1000 which are multiples of 2, 3, 5, and 7. Such a number must be a multiple of 210. Also, any multiple of 210 which is less than 1000 is in all four sets. Thus $\cap\mathcal{D} = \{210, 420, 630, 840\}$. □

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2. Collections of Functions

We may also consider sets whose members are functions.

Example 3. Let X be a set and let $\mathrm{Sym}(X)$ be the set of all bijective functions on X. Then $\mathrm{Sym}(X)$ is a collection of functions. $\square$

If A and B are sets, we may speak of the set of all functions from A to B. We shall denote this set by $\mathcal{F}(A, B)$:

$$\mathcal{F}(A, B) = \{f : A \to B\}.$$

Example 4. Let $A = \{1, 2\}$ and $B = \{5, 6, 7\}$. Then $\mathcal{F}(A, B)$ contains the following functions:

- $1 \mapsto 5$ and $2 \mapsto 5$;
- $1 \mapsto 5$ and $2 \mapsto 6$;
- $1 \mapsto 5$ and $2 \mapsto 7$;
- $1 \mapsto 6$ and $2 \mapsto 5$;
- $1 \mapsto 6$ and $2 \mapsto 6$;
- $1 \mapsto 6$ and $2 \mapsto 7$;
- $1 \mapsto 7$ and $2 \mapsto 5$;
- $1 \mapsto 7$ and $2 \mapsto 6$;
- $1 \mapsto 7$ and $2 \mapsto 7$.

Also $\mathcal{F}(B, A)$ contains the following functions:

- $5 \mapsto 1$, $6 \mapsto 1$, $7 \mapsto 1$;
- $5 \mapsto 1$, $6 \mapsto 1$, $7 \mapsto 2$;
- $5 \mapsto 1$, $6 \mapsto 2$, $7 \mapsto 1$;
- $5 \mapsto 1$, $6 \mapsto 2$, $7 \mapsto 2$;
- $5 \mapsto 2$, $6 \mapsto 1$, $7 \mapsto 1$;
- $5 \mapsto 2$, $6 \mapsto 1$, $7 \mapsto 2$;
- $5 \mapsto 2$, $6 \mapsto 2$, $7 \mapsto 1$;
- $5 \mapsto 2$, $6 \mapsto 2$, $7 \mapsto 2$.

$\square$

Example 5. Let $\mathcal{F} = \mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the set of all real valued functions of a real variable:

$$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}.$$

Let $\mathcal{D}$ denote the set of all differentiable functions in $\mathcal{F}$:

$$\mathcal{D} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}.$$

Note that $\mathcal{D} \subset \mathcal{F}$.

The differentiation operator is a function

$$\frac{d}{dx} : \mathcal{D} \to \mathcal{F}.$$

Not every function is the derivative of a function, so $\frac{d}{dx}$ is not surjective. Since two functions which differ by a constant have the same derivative, $\frac{d}{dx}$ is not injective. $\square$

3. Power Sets

Let X be a set. The *power set* of X is denoted $\mathcal{P}(X)$ and is defined to be the set of all subsets of X:

$$\mathcal{P}(X) = \{A \mid A \subset X\}.$$

Here are a few examples:

- $X = \varnothing \Rightarrow \mathcal{P}(X) = \{\varnothing\}$;
- $X = \{0\} \Rightarrow \mathcal{P}(X) = \{\varnothing, \{0\}\}$;
- $X = \{0,1\} \Rightarrow \mathcal{P}(X) = \{\varnothing, \{0\}, \{1\}, X\}$;
- $X = \{0,1,2\} \Rightarrow \mathcal{P}(X) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, X\}$.

and so forth. Here are some properties:

- $Y \subset X \Rightarrow \mathcal{P}(Y) \subset \mathcal{P}(X)$;
- $\cap\mathcal{P}(X) = \varnothing$;
- $\cup\mathcal{P}(X) = X$.

Let X be any set and let $T = \{0,1\}$. A given function $f : X \to T$ may be viewed as a subset of X by thinking of f as saying, for a given element, whether or not it is in the subset. The element 1 is thought of as "ON" or "TRUE" and the element 0 is thought of as "OFF" or "FALSE". Specifically, given $f : X \to T$, define A to the preimage of 1:

$$A = \{a \in A \mid f(a) = 1\};$$

that is, $A = f^{-1}[\{1\}]$.

On the other hand, given a subset of X, we can construct a function

$$\chi_A : X \to T$$

by defining

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in a. \end{cases}$$

This is just the characteristic function of the subset A.

Thus the power set of X corresponds to the set of functions from X into T in a natural way. Another way of stating this is that there exists a bijective function between $\mathcal{P}(X)$ and $\mathcal{F}(X,T)$.

4. Partitions

Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. We say that $\mathcal{C}$ *covers* X if $\cup\mathcal{C} = X$. We say that the sets in $\mathcal{C}$ are *collectively disjoint* if $\cap\mathcal{C} = \varnothing$. If for every two distinct sets $C, D \in \mathcal{C}$, we have $C \cap D = \varnothing$, we say that the members of $\mathcal{C}$ are *mutually disjoint* (or *pairwise disjoint*). If the sets of a collection are mutually disjoint, then they are collectively disjoint, but the converse of this is not necessarily true.

Example 6. Let $X = \{1,2,3\}$ and let $\mathcal{C} = \{\{1,2\},\{1,3\},\{2,3\}\}$. Then

$$\cup\mathcal{C} = (\{1,2\} \cup \{2,3\}) \cup \{2,3\} = \{1,2,3\} \cup \{2,3\} = \{1,2,3\} = X,$$

so the sets in $\mathcal{C}$ cover X. Also

$$\cap\mathcal{C} = (\{1,2\} \cap \{1,3\}) \cap \{2,3\} = \{1\} \cap \{2,3\} = \varnothing,$$

so the sets in $\mathcal{C}$ are collectively disjoint. They are not, however, mutually disjoint.

Let $\mathcal{D} = \{\{1,2\},\{3\}\}$. Then $\mathcal{D}$ covers X with mutually disjoint sets. □

A *partition* of X is a collection of mutually disjoint nonempty subsets of X which covers X. The members of a partition are called *blocks*.

Let $\mathcal{C} \subset \mathcal{P}(X)$. Then $\mathcal{C}$ is a partition of X if

(P0) $C \in \mathcal{C} \Rightarrow C \neq \varnothing$
(P1) $C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cap C_2 = \varnothing \vee C_1 = C_2$
(P2) $\cup\mathcal{C} = X$

Suppose that $\mathcal{C}$ is a partition of X. If $x \in X$, then there is a unique $A \in \mathcal{C}$ such that $x \in A$; x is certainly in one of them, because X is covered by the members of $\mathcal{C}$; x is in no more than one, for otherwise the ones containing x would overlap and not be disjoint. Put another way, every $x \in X$ is in exactly one of the members of $\mathcal{C}$.

Example 7. Let x be a point in a space and let $S(x,r)$ be a sphere of radius r with center x. Then the collection

$$\mathcal{S} = \{S(x,r) \mid r \in \mathbb{R} \text{ and } r \geq 0\}$$

is a partition of space; the blocks of this partition are spheres centered at x. This is true since each point in space has a unique distance from the point x. □

Example 8. Let C be the set of cards in a deck and let S be the set of suits. That is, C contains 52 elements and $S = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$. There is a natural function $f : C \to S$ which sends a given card to its suit. The preimage of a suit under f is the set of cards in that suit, for example:

$$f^{-1}[\spadesuit] = \{2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, \mathrm{J}\spadesuit, \mathrm{Q}\spadesuit, \mathrm{K}\spadesuit, \mathrm{A}\spadesuit\}.$$

Let $\mathcal{S} = \{f^{-1}[s] \mid s \in S\}$. Then $\mathcal{S}$ is a collection of subsets of C, each subset consisting of all the cards in a given suit. It is clear that $\mathcal{S}$ covers C and that the sets within $\mathcal{S}$ are mutually disjoint. Thus $\mathcal{S}$ is a partition of C. This is a general phenomenon: functions induce partitions on their domains. We will explore this in depth later.

One more thing to notice here. There are as many elements in $\mathcal{S}$ as there are in S. Indeed, in some philosophical way, $\mathcal{S}$ is *essentially the same* as the set S. □

5. Partitions Given by Functions

Let X and Y be sets and let $f : X \to Y$ be surjective. Let $y \in Y$. The *fiber over* y is $f^{-1}(y)$, the preimage of y.

The function f induces a partition of the set X is a natural way, by taking the collection of fibers. Let

$$\overline{X} = \{A \subset X \mid A = f^{-1}(y) \text{ for some } y \in Y\}.$$

Then $\overline{X}$ is a partition of X.

For each $x \in X$, let $\overline{x} = f^{-1}(f(x))$. Clearly, $\overline{x}$ is the fiber over $f(x)$, and $x \in \overline{x}$. That is, $\overline{x}$ is the set of elements in X which are mapped to $f(x)$ by f. As x ranges over all of X, we see that the blocks of the partition $\overline{X}$ are the subsets of X of the form $\overline{x}$. That is,

$$\overline{X} = \{A \in \mathcal{P}(X) \mid A = \overline{x} \text{ for some } x \in X\}.$$

The *canonical function* induced by f is

$$\beta : X \to \overline{X} \quad \text{given by} \quad \beta(x) = \overline{x}.$$

We use the greek letter β to remind us that this is the BAR function.

We attempt to define a function $\overline{f} : \overline{X} \to Y$ by setting $\overline{f}(\overline{x}) = f(x)$. The problem with this is that the definition appears to depend on which element in $\overline{x}$ we pick. If x' is another element in $\overline{x}$, then x' is in the fiber over $f(x)$, so $f(x') = f(x)$. Thus our definition does not depend on the particular element in $\overline{x}$ we pick; we say that the function $\overline{f}$ is "well-defined".

Proposition 1. *Let X and Y be sets and let $f : X \to Y$ be surjective. Let*

$$\overline{X} = \{A \subset X \mid A = f^{-1}(y) \text{ for some } y \in Y\}.$$

Let

$$\beta : X \to \overline{X} \quad \text{given by} \quad \beta(x) = \overline{x}.$$

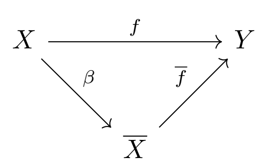
Set

$$\overline{f} : \overline{X} \to Y \quad \text{given by} \quad \overline{f}(\overline{x}) = f(x).$$

Then $\overline{f}$ is well-defined, and

$$\overline{f} \circ \beta = f.$$

This result is the *Isomorphism Theorem in the Category of Sets.* When we see the analogous theorem in other categories, it will be quite useful. The situation is represented by the following "commutative diagram".



Saying that the diagram commutes means that an element "flows" through the arrows from one set to another in a manner which does not depend on the route.

6. Exercises

Exercise 1. Design a collection $\mathcal{C}$ of subsets of $\mathbb{N}$ which has all of the following properties:

(1) $\mathcal{C}$ covers $\mathbb{N}$ ($\cup\mathcal{C} = \mathbb{N}$);
(2) distinct sets in $\mathcal{C}$ are disjoint ($C, D \in \mathcal{C}$ and $C \neq D \Rightarrow C \cap D = \varnothing$);
(3) each set $C \in \mathcal{C}$ contains infinitely many elements;
(4) $\mathcal{C}$ contains exactly 7 subsets of $\mathbb{N}$.

Recall that we have given the name "partition" to collections of sets satisfying the first two properties.

Exercise 2. Let $\mathbb{R}$ be the set of real numbers.
(a) Find a collection of subsets of $\mathbb{R}$ which covers $\mathbb{R}$ but whose members are not collectively disjoint.
(b) Find a collection of subsets of $\mathbb{R}$ which covers $\mathbb{R}$ and whose members are collectively disjoint but not mutually disjoint.
(c) Find three different partitions of $\mathbb{R}$, each containing a different number of blocks.

Exercise 3. Let $X = \{1, 2, 3, 4, 5\}$ and let $Y = \{1, 2, 3\}$. Find a five different partitions of the set $\mathcal{F}(X, Y)$, each of which contains three blocks.

Exercise 4. Let X be a set and let $A, B \subset X$.
(a) Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(b) Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.
(c) Find an example such that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

Exercise 5. Let X be a set. Find an injective function $\phi : X \to \mathcal{P}(X)$.

Exercise 6. Let X be as set. Show that there does not exist a surjective function $\phi : X \to \mathcal{P}(X)$.
(Hint: select an arbitrary function $\phi : X \to \mathcal{P}(X)$, and construct a set in $\mathcal{P}(X)$ which is not in the image of ϕ.)

Exercise 7. Let X be a set. Define a function $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $A \mapsto X \smallsetminus A$. Show that ϕ is bijective.

Exercise 8. Let X be a set and let $T = \{0, 1\}$. Show that there is a correspondence between the sets $\mathcal{P}(X)$ and $\mathcal{F}(X, T)$.

Exercise 9. Let X be a set containing n elements. Count the size of the set $\mathcal{P}(X)$.

Exercise 10. Let A and B be sets containing m and n elements respectively. Count the size of the set $\mathcal{F}(A, B)$.

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